ELSEVIER

Discrete Mathematics 216 (2000) 279–285

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

On hamiltonian connectedness of $K_{1,4}$-free graphs

MingChu Li

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3

Received 19 March 1997; revised 28 September 1999; accepted 14 October 1999

Abstract

G. Chen and R.H. Schelp proved that every 3-connected $K_{1,4}$-free graph G on n vertices with $\sum_{i=1}^{3} d(v_i) \geqslant n+4$ for any independent set $\{v_1, v_2, v_3\}$ of G is hamiltonian connected. In this paper, we show that every 3-connected $K_{1,4}$-free graph $G \notin J$ on at most $4\delta - 10$ vertices is hamiltonian connected, where J is the set of exceptional graphs.

Keywords: Hamiltonian connectedness; $K_{1,4}$-free graphs; Minimum degrees

1. Introduction

In this paper we deal with finite simple graphs. Let G be a graph. We denote by $\delta(G)$ (or δ) and $k(G)$ the minimum degree and the connectivity of G, respectively. A graph is called $K_{1,4}$-free if it does not contain a copy of $K_{1,4}$ as an induced subgraph. For a vertex v of G, the neighborhood $N(v)$ of v is the set of all vertices that are adjacent to v. For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G-H$ and $G[S]$ the induced subgraphs of G on $V(G)-V(H)$ and S, respectively. We denote by $N_H(S)$ the set of all vertices of H adjacent to some vertex of S, and let $N(S)=\bigcup_{x\in S} N(x)$ and $d_H(S)=|N_H(S)|$. For A and B in $V(G)$, let $E_G(A,B)=\{uv \in E(G)\colon u \in A \text{ and } v \in B\}$ and $e_G(A,B)=|E_G(A,B)|$. Let $P=x_1x_2\dots x_t$ be a path in G. Then uPv denotes the path $ux_1x_2\dots x_tv$ or $ux_tx_{t-1}\dots x_1v$. Let $x_i^+ = x_{i+1}$ for $1 \leqslant i < t$ and $x_i^- = x_{i-1}$ for $1 < i \leqslant t$, and let $x_iPx_j = P[x_i,x_j] = x_ix_{i+1}\dots x_j$, $x_jP^-x_i = P^-[x_j,x_i] = x_jx_{j-1}\dots x_i$ $(1 \leqslant i \leqslant j \leqslant t)$ and $P(x_i,x_j) = P[x_i,x_j] - \{x_i,x_j\}$. We will also consider $P(x_i,x_j)$ and $P[x_i,x_j]$ as the vertex sets. Let $x^{++}=(x^+)^+$ and $x^{--}=(x^-)^-$. We denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices of G. Other notation and terminology not defined here can be found in [1].

E-mail address: mingchu@math.toronto.edu (M. Li)

0012-365X/00/\$ - see front matter
PII: S0012-365X(99)00355-6

There have been many results in recent years dealing with hamiltonian cycles in graphs. O. Ore [5] showed the following result.

Theorem 1 (Ore [5]). *Let G be a graph of order n. If $\sigma_2(G) \geqslant n+1$, then G is hamiltonian connected.*

A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Matthews and Sumner [4] proved the following important result.

Theorem 2 (Matthews and Summner [4]). *Every 2-connected claw-free graph G on n vertices is hamiltonian if $n \leqslant 3\delta+2$.*

Recently, Markus [3] obtained a result similar to Theorem 2 for $K_{1,4}$-free graphs.

Theorem 3 (Markus [3]). *Every 2-connected $K_{1,4}$-free graph G on n vertices is hamiltonian if $n \leqslant 3\delta-2$.*

G. Chen and R.H. Schelp proved the following very interesting result.

Theorem 4 (Chen and Schelp [2]). *Every k-connected $K_{1,4}$-free graph of order n with $\sigma_k(G) \geqslant n+k+1$ is hamiltonian connected.*

From Theorem 4, we have the following corollary.

Theorem 5. *Every 3-connected $K_{1,4}$-free graph of order $n \leqslant 3\delta-4$ is hamiltonian connected.*

Let J be the set of all graphs defined as follows: Any graph G in J contains three disjoint hamiltonian connected subgraphs G_1, G_2 and G_3 such that

$$\bigcap_{i=1}^{3} N(V(G_i)) = \{u, v, x_2\}$$

and

$$E_G(G_i, G_j) = \emptyset \quad \text{for } i \neq j \text{ and } i, j = 1, 2, 3,$$

where $u \neq v \neq x_2 \neq u$ and $V(G) = \bigcup_{i=1}^{3} V(G_i) \cup \{u, v, x_2\}$ (see Fig. 1), or G contains three disjoint hamiltonian connected subgraphs G_1, G_2 and G_3 such that

$$\bigcap_{i=1}^{3} N(V(G_i)) = \{u, v\}$$

and

$$E_G(G_i, G_j) = \{u_i u_j\} \quad \text{for } i \neq j \text{ and } i, j = 1, 2, 3,$$

where $u \neq v$, $u_i \in V(G_i)$ for $i = 1, 2, 3$ and $V(G) = \bigcup_{i=1}^{3} V(G_i) \cup \{u, v\}$ (see Fig. 2).

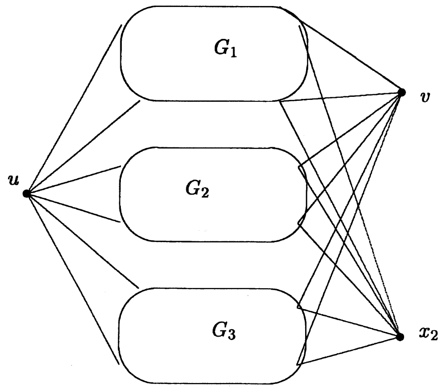

Fig. 1.

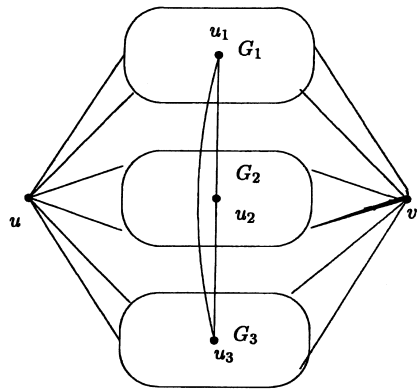

Fig. 2.

In this paper, we generalize Theorem 5 and prove the following theorem whose proof will be in Section 2.

Theorem 6. *Every* 3*-connected* $K_{1,4}$*-free graph* $G \notin J$ *of order* $n \leqslant 4\delta - 10$ *is hamiltonian connected.*

In the proof of Theorem 6, we use the following lemma whose proof is implicit in the proof of Theorem 8 of [2].

Lemma 7. *Let* G *be* 3*-connected* $K_{1,4}$*-free graph of order* n *which is not hamiltonian connected and* P *a longest non-hamiltonian path connecting* u *and* v *of* G, *and let* H *be a component of* $G - P$. *Then there exists an independent set* I *of* $d_P(H)$ *vertices in* G *such that*

$$\sum_{v \in I} d(v) \leqslant n + d_P(H).$$

2. Proof of Theorem 6

Assume that G is a graph of order n satisfying the conditions of Theorem 6. If the theorem is not true, then there are two vertices u and v in $V(G)$ such that there exists no hamiltonian path connecting u and v. Let $P=P[u,v]$ be a longest path connecting u and v, and H a component of $G-P$. We denote $N_P(H)=\{x_i: i=1,2,\ldots,t\}$, where $t=d_P(H)$. Obviously, $t\geqslant 3$ since G is 3-connected. We have the following claim using Lemma 7.

Claim 1. *For any component H of $G-P$, we have $d_P(H)=3$ and H contains at least $\delta-2$ vertices and $d_H(x)\geqslant\delta-3$ for any $x\in V(H)$.*

Proof. By Lemma 7, we obtain $d_P(H)\delta\leqslant n+d_P(H)$. Namely, $d_P(H)\leqslant n/(\delta-1)$. Since $n\leqslant 4\delta-10$ and G is 3-connected, we have $d_P(H)=3$. It is easy to see that $d_H(x)\geqslant\delta-3$ for any $x\in V(H)$ and $|V(H)|\geqslant\delta-2$.

We have from Claim 1 that $t=3$. Without loss of generality assume that $x_1=u$ and $x_3=v$ (and the proofs of other cases are similar to the following arguments). Let $S_i=P(x_i,x_{i+1})=x_i^+Px_{i+1}^-$ and $S_i^0=P(x_i^+,x_{i+1}^-)$ for $1\leqslant i\leqslant 2$ and $h=|V(H)|$. In order to prove this theorem, we need to verify the following 10 claims. We first have the following claim.

Claim 2. *For any component H of $G-P, H$ is hamiltonian connected.*

Proof. Otherwise, by Theorem 1, we know that there are two nonadjacent vertices u_1 and v_1 in H such that $|V(H)|\geqslant d_H(u_1)+d_H(v_1)\geqslant 2\delta-6$. By Lemma 7, there is an independent set of three vertices w_0, w_1 and w_2 in G such that $d(w_0)+d(w_1)+d(w_2)\leqslant n+d_P(H)$. From the proofs of Theorems 7 and 8 in [2, pp. 431–435], we can assume that w_1 and w_2 are on P and w_0 is on H, and we also know that w_i does not belong to $N_P(H)$ and then has no neighbors in H for $i=1,2$. Thus the inequality above can be improved as

$$d(w_1)+d(w_2)+d(w_0)+|V(H)-(N_H(w_0)\cup\{w_0\})|\leqslant n+d_P(H).$$

Namely, $d(w_1)+d(w_2)+|V(H)|\leqslant n+d_P(H)$. So $n\geqslant 2\delta+2\delta-9=4\delta-9$, a contradiction. $\square$

Note that any pair of vertices x_i, $x_j\in N_P(H)$ is joined by a path with all internal vertices in H. We denote such a path by x_iHx_j for $i\neq j$.

Claim 3. *$G-P$ has only one component H.*

Proof. Otherwise, let H' and H be two distinct components of $G-P$. Then we have that H and H' contain at least $\delta-2$ vertices. The path $uHx_2P[x_2^+,v]$ gives

$a=|P(u,x_2)|\geqslant h$. Similarly, $b=|P(x_2,v)|\geqslant h$. Hence we obtain $n\geqslant|V(P)|+|V(H)|+|V(H')|\geqslant 3h+|V(H')|+3\geqslant 4\delta-5$, a contradiction.

Claim 4. *$N(x_2^+)-(\{x_2^-\}\cup N_P(H))$ is contained in S_2^0, and $N(x_2^-)-(\{x_2^+\}\cup N_P(H))$ is contained in S_1^0.*

Proof. Suppose that w belongs to $(N_P(x_2^+)-(\{x_2^-\}\cup N_P(H)))\cap S_1^0$. Then the path

$$uHx_2P^-[x_2^-,w]P[x_2^+,v]$$

gives $|P(u,w)|\geqslant h$.

Consider the vertex w^+. Let u_2 be the neighbor of w^+ closest to u on $P[u^+,w]$. Then the path

$$uHx_2P^-[x_2^-,w^+]P[u_2,w]P[x_2^+,v]$$

gives $a=|p(u,u_2)|\geqslant h$. If $N(w^+)\cap S_2\neq\emptyset$, let u_1 be the neighbor of w^+ closest to v on S_2. Then the path

$$P[u,w]P[x_2^+,u_1]P[w^+,x_2]Hv$$

gives $b=|P(u_1,v)|\geqslant h$. If $N(w^+)\cap S_2=\emptyset$, then let $u_1=x_2$ and $b=|P(x_2,v)|$. Thus $N(w^+)\cup\{w^+\}$ is contained in $P[u_2,u_1]=B$. Hence $|B|\geqslant\delta+1$ and $n\geqslant a+b+|B|+h\geqslant 4\delta-5$. This contradiction shows that $N(x_2^+)-(\{x_2^-\}\cup N_P(H))$ is contained in S_2^0. Similarly, $N(x_2^-)-(\{x_2^+\}\cup N_P(H))$ is contained in S_1^0. Thus Claim 4 is true. □

In the following, let $f\in N(x_2^+)$ be the closest vertex to v on $P(x_2^+,v)$ and $g\in N(x_2^-)$ the closest vertex to u on $P(u,x_2^-)$. Then we have the following fact.

Claim 5. *$N(u^+)$ and $P(x_2^+,f)$ are disjoint, and $N(v^-)$ and $P(g,x_2^-)$ are disjoint.*

Proof. Suppose, to the contrary, that $u^+w\in E(G)$ and $w\in P(x_2^+,f)$. Let f' be the neighbor of x_2^+ closest to w on $P(w,f)$. Then $N(x_2^+)\cup\{x_2^+\}$ is contained in $P[x_2^-,w]\cup P[f',f]\cup\{u,v\}=B$. The path

$$uHx_2P^-[x_2^-,u^+]P^-[w,x_2^+]P[f',v]$$

gives $a=|P(w,f')|\geqslant h$. Hence $|V(P)|\geqslant a+d(x_2^-)+|B|-4\geqslant 3\delta-5$, and so $n\geqslant 4\delta-7$. This contradiction shows $N(u^+)\cap P(x_2^+,f)=\emptyset$. Similarly, $N(v^-)\cap P(g,x_2^-)=\emptyset$. Hence we complete the proof of Claim 5. □

Claim 6. *$N(u^+)\cap P(x_2,v^-)=\emptyset$, and $N(v^-)\cap P(u^+,x_2)=\emptyset$.*

Proof. Assume that $wu^+\in E(G)$ and $w\in P(x_2,v^-)$. Then we have from Claim 5 that w must be in $P[f,v^-]$. Furthermore, we have the following fact.

$$N(v^-)\cap P(x_2^+,f)=\emptyset.$$

Otherwise, let $xv^- \in E(G)$ and $x \in P(x_2^+, f)$ and f' be the neighbor of x_2^+ closest to x on $P(x, f)$. Then $N(x_2^+) \cup \{x_2^+\}$ is contained in $P[x_2^-, x] \cup P[f', f] \cup \{v, u\} = B$. The path

$$P' = uHx_2P^-[x_2^-, u^+]P^-[w, f']P[x_2^+, x]v^-v$$

gives $a = |P(x, f')| + |P(w, v^-)| \geqslant h$. Hence $n \geqslant |V(P)| + |V(H)| \geqslant a + |B| + d(x_2^-) - 4 + h \geqslant 4\delta - 7$. This contradiction shows that the above fact is true.

Thus, by Claim 5, we have that $N(v^-) \cup \{v^-\}$ is contained in $P[f, v] \cup P[u, g] = D$. So we obtain that $n \geqslant |V(P)| + |V(H)| \geqslant |D| + d(x_2^-) + d(x_2^+) - 7 + h \geqslant 4\delta - 8$. This contradicton shows the completion of the proof of Claim 6. □

In the following, let us assume that x is the neighbor of u^+ closest to x_2 on S_1 and y the neighbor of v^- closest to x_2 on S_2. Then, using Claim 6 we have the following fact.

Claim 7. *$x \in P(g, x_2)$ and $y \in P(x_2, f)$.*

Proof. Suppose, to the contrary, that $x \notin P(g, x_2)$. Then, by Claim 6, $x \notin P(x_2, v^-)$. Thus $x \in P[u^+, g]$. It follows from Claim 4 that $a = |S_1| = |P[u^+, g]| + |P(g, x_2)| \geqslant d(u^+) + d(x_2^-) - d_P(H) - 3 \geqslant 2\delta - 6$. Hence $|V(P)| \geqslant |S_1| + |S_2| + |\{u, x_2, v\}| \geqslant 3\delta - 6$ and so $n \geqslant 4\delta - 8$. This contradiction shows that Claim 7 is proved.

Furthermore, we have also the following claim.

Claim 8. *$E_G(P(u^+, x_2^-), P(x_2^+, v^-)) = \emptyset$.*

Proof. We obtain from Claim 7 that for any vertex $w \in P(u^+, x_2^-)$, we have $w \in P(u^+, x)$ or $w \in P(g, x_2^-)$, and for any vertex $z \in P(x_2^+, v^-)$, we have $z \in P(x_2^+, f)$ or $z \in P(y, v^-)$. Suppose that $w_1 \in P(u^+, x_2^-)$ and $w_2 \in P(x_2^+, v^-)$ such that $w_1w_2 \in E(G)$. Without loss of generality, assume that $w_1 \in P(u^+, g)$ and $w_2 \in P(x_2^+, f)$ (and the proofs of other cases are similar). Let f' be the neighbor of x_2^+ closest to w_2 on $P(w_2, f)$. Then $N(x_2^+) \cup \{x_2\}$ is contained in $P[x_2^-, w_2] \cup P[f', f] \cup \{u, v\} = B$. The path

$$uHx_2P^-[x_2^-, w_1]P^-[w_2, x_2^+]P[f', v]$$

gives $a = |P(u^+, w_1)| + |P(w_2, f')| \geqslant h$. Hence $n \geqslant a + |B| + |P[g, x_2^-]| - 1 + h \geqslant 4\delta - 7$. This contradiction shows the completion of our proof of Claim 8. □

Claim 9. *$u^+v^- \notin E(G)$.*

Proof. Suppose, to the contrary, that $u^+v^- \in E(G)$. Since G is 3-connected, by Claim 8, we know that at least one of $\{u, v\}$ must have neighbors in $P(u^+, v^-)$. Without loss of generality, assume that $wv \in E(G)$ and $w \in P(g, x_2^-)$ (and the proofs of other cases are similar), and let g' be the neighbor of x_2^- closest to w on $P(g, w)$. Then $N(x_2^-) \cup \{x_2^-\}$ is contained in $P[g, g'] \cup P[w, x_2^-] \cup \{u, v\} = B$. The path

$$uHx_2P[x_2^+, v^-]P[u^+, g']P^-[x_2^-, w]v$$

gives $a=|P(g',w)|\geqslant h$. Hence $n\geqslant a+|B|+d(x_2^+)-3+h\geqslant 4\delta-6$. This contradiction shows the completion of the proof of Claim 9.

Claim 10. *$G[S_1]$ and $G[S_2]$ are hamiltonian connected.*

Proof. We easily know from Claims 7 and 8 that for any vertex $z\in S_1$, z has at most three neighbors outside of S_1. Hence we have $d_{S_1}(z)\geqslant\delta-3$. Since $n\leqslant 4\delta-8$ and $h\geqslant\delta-2$, $|P[u,v]|\leqslant 3\delta-6$. Thus, $|S_1|\leqslant 2\delta-7$ since otherwise $|P[u,v]|\geqslant|S_1|+|S_2|+3\geqslant 3\delta-5$. So $d_{S_1}(z)\geqslant(|S_1|+1)/2$. It follows from Theorem 1 that $G[S_1]$ is hamiltonian connected. Similarly, $G[S_2]$ is hamiltonian connected.

Claim 11. *If $x_2^-x_2^+\in E(G)$, then $N_P(x_2)-\{x_2^+,x_2^-,u,v\}=\emptyset$.*

Proof. Let us assume that $u'\in N_P(x_2)-\{x_2^+,x_2^-,u,v\}$ and $u'\in S_2$. Since $G[S_2]$ is hamiltonian connected (by Claim 10), there is a hamiltonian path $Q=Q[x_2^+,u']$ connecting x_2^+ and u' in $G[S_2]$. It follows that G has a hamiltonian path $P'=P[u,x_2^-]Q[x_2^+,u']x_2Hv$, a contradiction.

We now complete the proof of Theorem 6. □

Since G is 3-connected, by Claims 1–11, we know that $N(u)\cap S_2\neq\emptyset$ and $N(v)\cap S_1\neq\emptyset$. If $x_2^-x_2^+\notin E(G)$, it is easy to see that G yields the graph in Fig. 1. Thus $x_2^-x_2^+\in E(G)$. Consider the graph $G_1=G[V(H)\cup\{x_2\}]$. By Claim 11, $d_{G_1}(x_2)\geqslant\delta-4$. For any vertex w in H, we have that $d_{G_1}(w)\geqslant\delta-2$ since $d_P(H)=3$. Thus $\delta(G_1)\geqslant\delta-4$. Since $|P[u,x_2^-]|\geqslant d(u^+)$ and $|P[x_2^+,v]|\geqslant d(v^-)$ by Claims 5,9 and 11, $|V(G_1)|\leqslant n-2\delta\leqslant 2\delta-10$. Thus $\delta(G_1)\geqslant(|V(G_1)|+2)/2$. By Theorem 1, G_1 is hamiltonian connected. It follows that G yields the graph in Fig. 2. Therefore the proof of Theorem 6 is completed. □

Acknowledgements

The author would like to thank Professor Derek Corneil and Professor Eric Mendelsohn for their useful suggestions and the referee for his useful suggestions and careful corrects to the manuscript.

References

[1] J.A. Bondy, U.S. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[2] G. Chen, R.H. Schelp, Hamiltonicity for $K_{1,r}$-free graphs, J. Graph Theory 20 (1995) 423–439.
[3] L.R. Markus, Hamiltonian results in $K_{1,r}$-free graphs, Congr. Numer. 98 (1993) 143–149.
[4] M. Matthews, D. Sumner, Longest cycles and paths in claw-free graphs, J. Graph Theory 9 (1985) 269–277.
[5] O. Ore, Hamiltonian connected graphs, J. Math. Pure Appl. 42 (1963) 21–27.